Eulerian-Lagrangian formulation for flow in soils: Theory

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An Eulerian scheme is used to reformulate the equations of groundwater with pressure dependent density flowing through saturated zones, and moisture transport in unsaturated zones. The governing equation is decomposed into advection along characteristic path lines and propagation of the residue at a fixed frame of reference. This formal decomposition enables the handling of physical phenomena that incorporate discontinuities and/or steep gradient without the need for means of artificial smoothing.

INTRODUCTION

Most conventional methods for parabolic conduction/flow problems are classified into an Euler representation for the distribution of state variables in time and space [1]–[9]. The numerical implementation of such a concept evolves a discretization of a fixed frame of reference into finite subdomains.

In recent developments dealing with advection-dispersion phenomena it was proposed to reformulate the governing equation with regard to Euler–Lagrange concepts for describing motion [10]–[12]. Such methods are inherently suited for tracking sharp fronts and discontinuities propagating in time and space. Also, when implementing to numerical schemes this method combines the simplicity of the fixed Eulerian grid with the computational power of the Lagrangian approach.

This paper describes an alternate formulation of motion in soil, based on the mixed Eulerian–Lagrangian method. It is believed that adoption of the novel approach for flow problems, especially those that introduce numerical difficulties (e.g. two phase flow, steep gradients of the state variables, etc.) will yield better predictions.

PRESSURE DEPENDENT DENSITY FLUID IN SATURATED ZONES

A. Governing equation

The mass balance equation governing compressible groundwater flowing through an homogeneous porous medium, without sources or sinks, neglecting time changes corresponding to different species and dispersion or diffusion of the mass, dictates

$$n\frac{\partial\gamma}{\partial t}=-\nabla(\gamma\mathbf{q}) \tag{1}$$

Where, n = constant effective porosity, $\mathbf{q}$ = Darcy's velocity vector, γ = groundwater specific weight. Being pressure dependent, the left side of (1) is expanded to

$$n\frac{\partial\gamma}{\partial t}=n\gamma\beta\frac{\partial P}{\partial t} \tag{2a}$$

$$\beta=\frac{1}{\gamma}\frac{\partial\gamma}{\partial P} \tag{2b}$$

Here, β = compressibility coefficient, P = fluid pressure. Following Hubbert [2], the hydraulic head (h) is defined as

$$h=Z^*+\int_{P_a}^{P}\frac{\mathrm{d}P}{\gamma} \tag{3}$$

Where, Z^* = the elevation above a prescribed datum, Pa = atmospheric pressure.

If we differentiate equation (3) with respect to spatial coordinates $\mathbf{X}$, and time t, we obtain,

$$\nabla h=\nabla Z^*+\frac{1}{\gamma}\nabla P \tag{4}$$

$$\frac{\partial h}{\partial t}=\frac{1}{\gamma}\frac{\partial P}{\partial t} \tag{5}$$

Darcy's Law relates fluid's specific flux with its pressure head and elevation by the following

$$\mathbf{q}=-\frac{\mathbf{k}\gamma}{\mu}\left(\frac{\nabla P}{\gamma}+\nabla Z^*\right) \tag{6}$$

Where, μ = dynamic viscosity, $\mathbf{k}$ = the intrinsic permeability tensor.

Thus by virtue of equation (4) equation (6) is rewritten to

$$\mathbf{q}=-\frac{\mathbf{k}\gamma}{\mu}\nabla h \tag{7}$$

By virtue of equation (2) and (5) γ and h are related by

$$\frac{\partial\gamma}{\partial t}=\beta\gamma^2\frac{\partial h}{\partial t} \tag{8}$$

In view of (1), (7) and (8) we obtain

$$\nabla(\mathbf{K}^*\nabla h)=S^*\frac{\partial h}{\partial t} \tag{9}$$

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$$\mathbf{K}^* = \frac{\mathbf{k}\gamma^2}{\mu} \tag{10}$$

$$S^* = n\beta\gamma^2 \tag{11}$$

subject to,

Initial Condition (Cauchy)

$$h(\mathbf{X},o) = ho(\mathbf{X}) \quad \text{on} \quad \mathbf{X}\,E\,R, t=0 \tag{12}$$

Boundary Condition

$$-\mathbf{K}^*\nabla h \cdot \hat{n} + \alpha_\Gamma(h - h_\Gamma) = \mathbf{q}_\Gamma^* \quad \text{on} \quad \mathbf{X}\,E\,\Gamma \times (o,\tau] \tag{13}$$

Where R is the region of interest enclosed by the boundary Γ with $\hat{n}$ normal unit outward vector.

Also, $ho(\mathbf{X})$, $h_\Gamma(\mathbf{X},t)$, $\mathbf{q}_\Gamma = \gamma \mathbf{q}_\Gamma(\mathbf{X},t)$ are prescribed functions and $\alpha_\Gamma(t)$ controls the type of condition prevailing at $\mathbf{X}\,E\,\Gamma$.

Dirichlet Condition

$$\alpha \to \infty \tag{14a}$$

$$h = h_\Gamma \tag{14b}$$

Neumann Condition

$$\alpha \to o \tag{14e}$$

$$-\mathbf{K}^*\nabla h \cdot \hat{n} = \mathbf{q}_\Gamma^* \tag{14d}$$

Mixed Condition

$$o < \alpha < \infty \tag{15}$$

B. Eulerian–Lagrangian method

Define

$$S^* = 1 + s^* \tag{16a}$$

and a time derivative of the form

$$\frac{L}{Lt} = \frac{\partial}{\partial t} - (\nabla \mathbf{k}^*) \cdot \nabla \tag{16b}$$

Thus when substituting (16) into (10) the form resulting is

$$\frac{Lh}{Lt} + s^* \frac{\partial h}{\partial t} = \mathbf{K}^* \nabla^2 h \tag{17}$$

It is noteworthy to notice that equation (17), unlike (9), describes the evolution of h moving along characteristic lines defined by the path equation.

$$\frac{L\mathbf{X}}{Lt} = -\nabla \mathbf{K}^* \tag{18}$$

Let us now decompose h into two parts

$$h = \bar{h} + {}^\circ h \tag{19}$$

One way to perform such a decomposition is to let $\bar{h}$ satisfy

$$\frac{L\bar{h}}{Lt} = 0 \tag{20}$$

subject to,

$$\bar{h}(\mathbf{X},0) = ho(\mathbf{X}) \quad \text{on } \mathbf{X}\,E\,R,\ t=0 \tag{21}$$

$$\alpha_\Gamma(\bar{h} - h_\Gamma) = \mathbf{q}_\Gamma^* \quad \text{on } \mathbf{X}\,E\,\Gamma \times (o,\tau] \tag{22}$$

Clearly, the $\bar{h}$ value, while being conveyed through the field, remains constant during time evolution. The residue ${}^\circ h$ must then satisfy the equation

$$\frac{L^\circ h}{Lt} + s^* \frac{\partial h}{\partial t} = \mathbf{K}^* \nabla^2 h \tag{23}$$

subject to,

$${}^\circ h(\mathbf{X},o) = 0 \quad \text{on} \quad \mathbf{X}\,E\,R, t=0 \tag{24a}$$

$$\alpha_\Gamma h - \mathbf{K}^*\nabla h \cdot \hat{n} = 0 \quad \text{on } X\ E\ \Gamma X(o,\tau] \tag{24b}$$

Equation (17) is thus rigorously decomposed to a set of two equations solved separately. The approach is to first solve the problem for $\bar{h}$ and then solve the residual ${}^\circ h$, with source terms generated by the first problem on the fixed grid. The composition of ${}^\circ h$ and $\bar{h}$ at any point in space and time is thus determined.

MOISTURE TRANSPORT IN UNSATURATED ZONE

It is intended to develop a similar alternative approach for the distribution of water pressure head ψ in soils. This procedure will be effective especially at existing steep gradients of ψ when compared to the hydraulic conductivity k_w tensor.

Assuming that the air is stationary (or its movements could be ignored comparably), in view of Darcy's law for constant water specific weight we have

$$\mathbf{q}_w = -K_w(\theta_w)\nabla(\psi + Z^*) \tag{25a}$$

$$K_w(\theta_w) = \frac{k_w(\theta_w)\gamma}{\mu} \tag{25b}$$

$$\psi = \frac{P}{\gamma} \tag{25c}$$

here, θ_w is the volumetric water content.

Assuming no sources or sinks, neglecting volumetric air content, mass conservation yields

$$\frac{\partial \theta_w}{\partial t} = -\nabla \cdot \mathbf{q}_w \tag{26}$$

For an homogeneous soil ($\theta_w = \theta_w(\psi)$, $K_w = K_w(\theta_w)$) in view of (25) and (26) we obtain

$$C_w \frac{\partial \psi}{\partial t} = \nabla \cdot (K_w \nabla \psi) + \nabla \cdot (K_w \nabla Z^*) \tag{27a}$$

Where the specific moisture capacity C_w is given by

$$C_w(\psi) = \frac{d\theta}{d\psi} w \tag{27b}$$

Equation (27a) is subject to

$$\psi(\mathbf{X},o) = \psi_0(\mathbf{X}) \quad \text{on} \quad \mathbf{X}\,E\,R,\ t=0 \tag{28}$$

$$-K_w \nabla\psi \cdot \hat{n} + \alpha_\Gamma(\psi - \psi_\Gamma) = Q_\Gamma \quad \text{on } X \in \Gamma \times (0,\tau] \quad (29)$$

By interchanging $\mathbf{K}^*$, h, h_0, h_Γ, $\mathbf{q}_\Gamma^*$ respectively with K_w, ψ, ψ_0, ψ_Γ, Q_Γ, the context given from equation (13) to equation (15) is adopted accordingly.

Furthermore if C_w is exchanged with S^* and by decomposing

$$\psi = \bar{\psi} + \mathring{\psi} \quad (30)$$

then by following the procedure described from equation (16) to (24), the set of equations to be solved are

$$\frac{L\bar{\psi}}{Lt} = 0 \quad (31)$$

subject to

$$\bar{\psi}(\mathbf{X},o) = \psi_0(\mathbf{X}) \quad \text{on} \quad \mathbf{X} \in R,\ t=0 \quad (32a)$$

$$\alpha_\Gamma(\bar{\psi} - \psi_\Gamma) = Q_\Gamma \quad \text{on} \quad \mathbf{X} \in \Gamma \times (0,\tau] \quad (32b)$$

and

$$\frac{L\mathring{\psi}}{Lt} + C_w \frac{\partial\psi}{\partial t} = K_w \nabla^2 \psi + \nabla \cdot (K_W \nabla Z^*) \quad (33)$$

subject to

$$\mathring{\psi}(\mathbf{X},0) = 0 \quad \text{on} \quad \mathbf{X} \in R,\ t=0 \quad (34a)$$

$$\alpha_\Gamma \mathring{\psi} - K_w \nabla \mathring{\psi} \cdot \hat{n} = 0 \qquad \mathbf{X} \in \Gamma \times (0,\tau] \quad (34b)$$

As before the solution of the problem for $\bar{\psi}$ initiates source terms for the solution of $\mathring{\psi}$.

CONCLUSION

The main contribution of the proposed reformulation of the flow equations is to attain a better reliable model to predict the propagation of discontinuities and sharp fronts associated with the state variables.

Singularities are imposed on the decomposed part of the state variable solved separately along characteristics. Thus no means of numerical smoothing are needed.

If a numerical implementation takes place, the Lagrangian method yields symmetric global matrices combined with the simplicity of the fixed Eulerian grid, all of which posseses very attractive features.

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